Introduction to GRMHD

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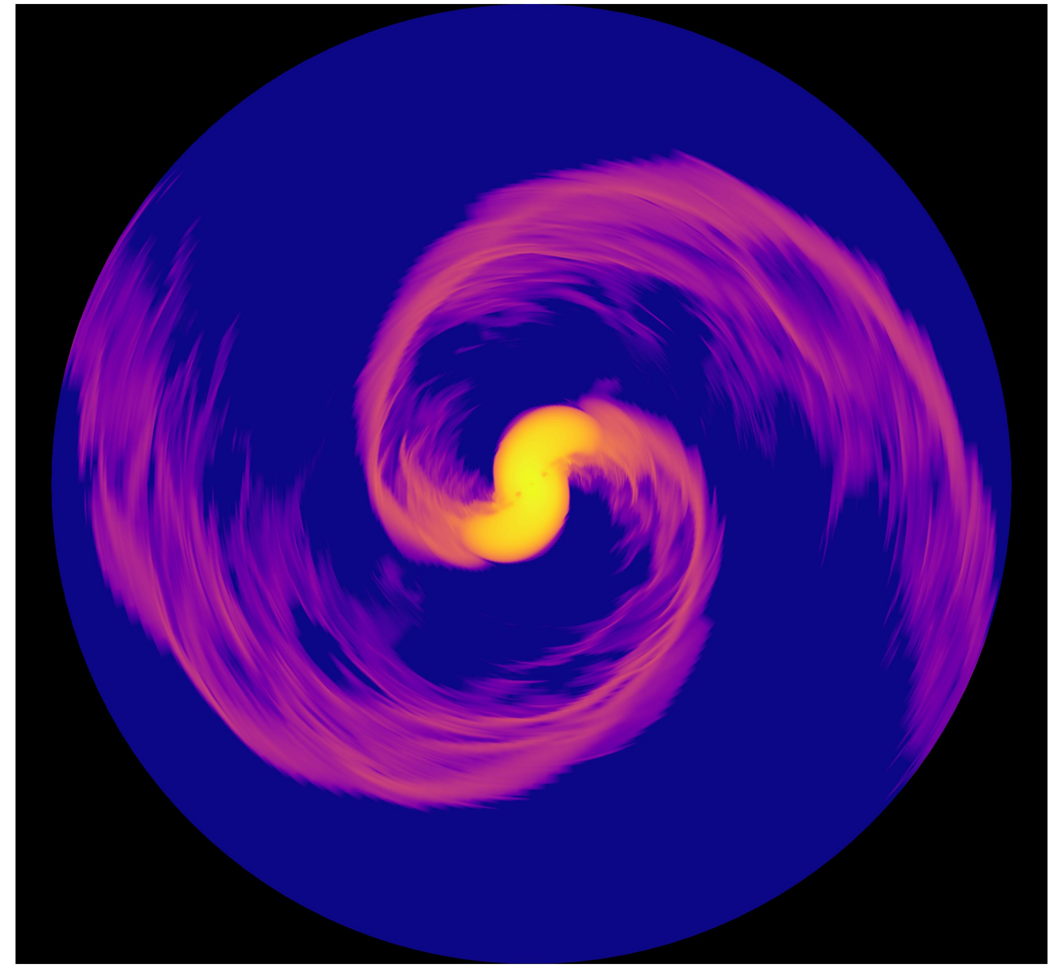


North American Einstein Toolkit School 2021

July 28 2021

Overview

- Introduction
- From **conservation laws** to
- The **Valencia formulation of GRMHD**, a first order flux-conservative hyperbolic system
- Building blocks of a **finite volume scheme** to solve the equations of GRMHD numerically
- Taking a look at an actual implementation: **GRHydro**

Textbooks and resources

- J. A. Font: Living Reviews in Relativity 11 (1), 1-13
- J. M. Martí and E. Müller: Living Reviews in Relativity 6, 7 (2003)
- L. Rezzolla and O. Zanotti: Relativistic Hydrodynamics
- T. W. Baumgarte and S. L. Shapiro: Numerical Relativity: Solving Einstein's Equations on the Computer
- M. Alcubierre: Introduction to 3+1 Numerical Relativity
- R. J. Leveque: Numerical Methods for Conservation Laws, Finite Volume Methods for Hyperbolic Problems
- F. Banyuls et al 1997 ApJ 476 221
- L. Antón et al 2006 ApJ 637 296

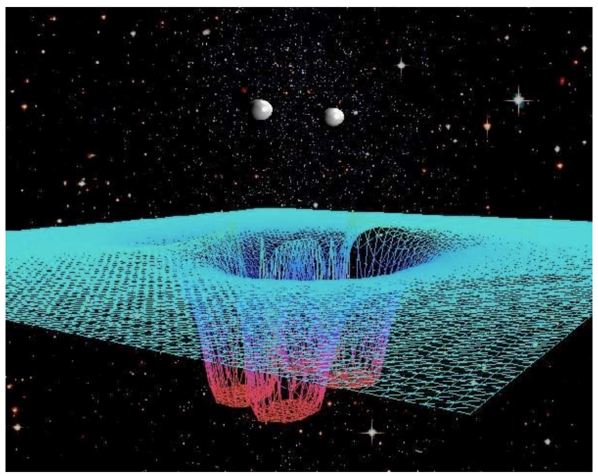
GW and EM emission

LIGO-VIRGO

Mass

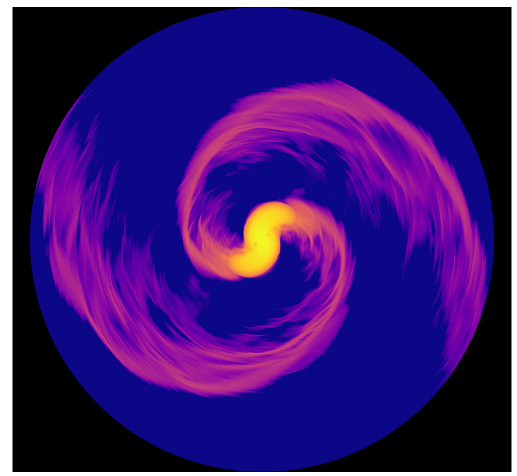
LISA, PTA

BBH
EM dark(?)

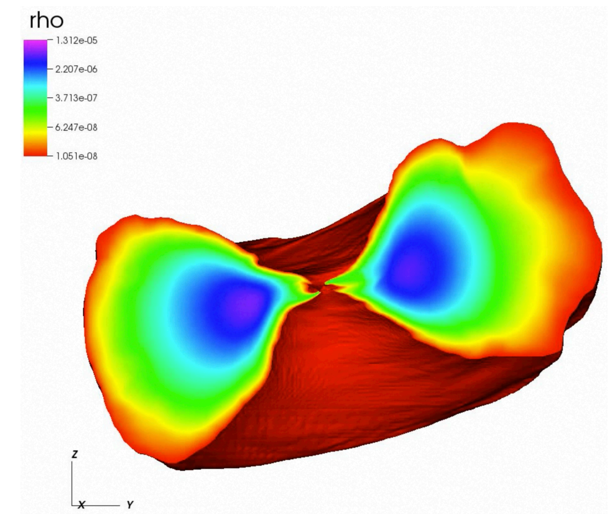


[Image credit: CCRG]

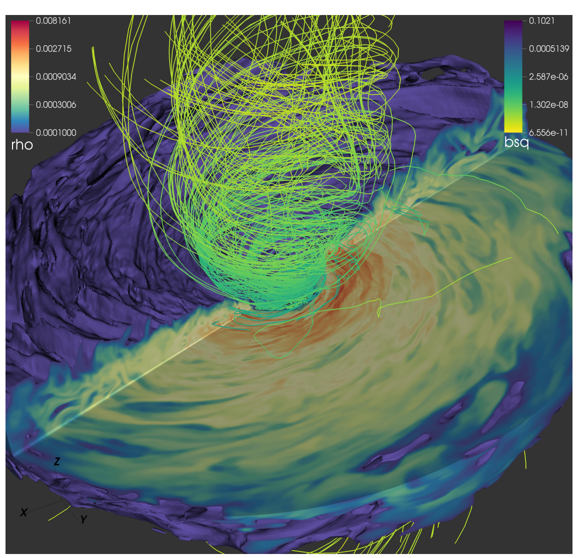
BNS
EM bright



BHNS
EM bright



SMBBH
EM bright



Supernovae are EM, neutrino, and GW bright!

Multi-messenger astronomy requires multi-physics simulations!

Gravitational waves - General (numerical) relativity, nuclear physics and thermodynamics (equation of state), self-gravitating matter

Electromagnetic radiation - Jets (general relativistic magnetohydrodynamics, GR force-free electrodynamics), radiation physics (radiation hydrodynamics)

Neutrinos - nuclear physics, neutrino transport (neutrino leakage, truncated moment radiation hydrodynamics) mixed elliptic-hyperbolic PDE

conservation laws (for matter)

$$\boxed{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R} = 8\pi \boxed{T_{\mu\nu}}$$

Ricci tensor **metric tensor** **Ricci scalar** **stress-energy tensor**

In numerical relativity, we are solving the Einstein equations coupled to conservation laws for the matter sources. These are a set of highly **non-linear partial differential equations coupling** the evolution of the **4D spacetime** to its **energy and momentum content** (matter and the gravitational field itself).

Which **matter sources** are **dynamically important** for the spacetime?

Binary neutron star/neutron star black hole mergers and their postmerger remnants, supernovae, gravitational waves from these systems.

Gravitational collapse.

The current state of the art of numerical relativity matter simulations

Numerical relativity: BSSN/CCZ4 (McLachlan, BAM, SACRA, BSSN_MoL), pseudospectral codes (SpEC), DG methods (Spectre), Generalized Harmonic formulation (HAD)

GRMHD: High resolution shock capturing methods for hyperbolic conservation laws, constrained transport, constraint damping, or A-field evolution for the div.B=0 constraint (GRHydro, Whisky, WhiskyTHC, BAM, SpEC, IllinoisGRMHD, HAD, Kyoto Code, Harm3D, Spectre, Cosmos++)

EOS: realistic equations of state beyond polytropes and ideal gases: inclusion of microphysical EOS that are temperature and constitution-dependent (LS, SFHo, SLy, FPS, APR, Shen)

Neutrinos: leakage schemes (Whisky, WhiskyTHC, SpEC, Kyoto Code) and truncated moment formalism (AGILE-Boltztran, Chimera, thornado, GENASIS, WhiskyTHC, SpEC, Kyoto Code) to approximate neutrino cooling and heating

Breakthrough matter simulations in numerical relativity

- 2000: First GRHD BNS merger simulation [Shibata and Uryu 2000]
- 2006: First GRHD BHNS merger simulation [Shibata and Uryu 2006]
- 2008: First GRMHD BNS merger simulations [Anderson et al 2008, Liu et al 2008]
- 2010: First GRMHD BHNS merger simulation [Chawla et al 2010]

4D Conservation laws in GRMHD

$\nabla_\mu(\rho u^\mu) = 0$ continuity equation,

$\nabla_\mu T^{\mu\nu} = \nabla_\mu(T^{\mu\nu}_{\text{matter}} + T^{\mu\nu}_{\text{EM}}) = 0$ conservation of energy-momentum,

$\nabla_\nu(*F^{\mu\nu}) = 0$ Maxwell's equations,

$P = P(\rho, \epsilon), P(\rho, T, Y_e)$ equations of state.

∇_μ is the covariant derivative compatible with the spacetime metric $g_{\mu\nu}$, ρ is the fluid rest-mass density, u^μ is the fluid four-velocity, $T^{\mu\nu}$ is the stress-energy tensor, and $*F^{\mu\nu}$ is the dual of the Maxwell tensor, respectively.

See **Josh's lecture** for more material on the **EOS**.

Let's start by obtaining the **stress-energy tensors**…

In all that follows, we will make **two assumptions** about our fluids:

Neglecting non-adiabatic effects (**perfect fluid**) and vanishing resistivity (the **ideal MHD** limit)

Starting with $T^{\mu\nu}_{\rm EM}$, we express it in terms of the Faraday tensor:

$$T^{\mu\nu}_{\rm EM} = F^{\mu\lambda}F^{\nu}_{\lambda} - \frac{1}{4}g^{\mu\nu}F^{\lambda\kappa}F_{\lambda\kappa}\,,$$

and decompose the Faraday tensor as

$F^{\mu\nu} = U^{\mu}E^{\nu}_{(U)} - U^{\nu}E^{\mu}_{(U)} + \epsilon^{\mu\nu\lambda\kappa}U_{\lambda}B_{\kappa(U)}$, where $\epsilon^{\mu\nu\lambda\kappa} \equiv \frac{-1}{\sqrt{-g}}[\mu\nu\lambda\kappa]$ and $[\mu\nu\lambda\kappa]$ and $\sqrt{-g}$ are the totally antisymmetric Levi-Civita symbol and determinant of the spacetime metric, respectively.

$E^{\mu}_{(U)}$ and $B^{\mu}_{(U)}$ are the electric and magnetic fields measured by an observer with a generic four-velocity U^{μ} and are given by:
$E^{\mu}_{(U)} = F^{\mu\nu}U_{\nu}$ and $B^{\mu}_{(U)} = \frac{1}{2}\epsilon^{\mu\nu\kappa\rho}U_{\nu}F_{\rho\kappa}$.

Both electric and magnetic fields are orthogonal to U^{μ} : $E^{\mu}_{(U)}U_{\mu} = B^{\mu}_{(U)}U_{\mu} = 0.$

In the following, we will focus on two particular observers: Observers comoving with the fluid having four-velocity u^μ and normal observers with four-velocity

$$n^\mu = \left(\frac{1}{\alpha}, -\frac{\beta^i}{\alpha}\right), n_\mu = (-\alpha, 0),$$

and denote the electric and magnetic fields measured by them as $E^\mu_{(u)}, B^\mu_{(u)}$ and E^μ, B^μ, respectively.

In the so-called **ideal MHD limit** (the limit of **vanishing resistivity** or **infinite conductivity**), the electric field measured by an observer comoving with the fluid vanishes:

$$E^\mu_{(u)} = F^{\mu\nu} u_\nu = 0,$$

and the Faraday tensor and its dual can be expressed solely in terms of the magnetic field measured by the comoving observer:

$$F^{\mu\nu} = \epsilon^{\mu\nu\lambda\kappa} u_\lambda B^{(u)}_\kappa \text{ and } {}^*F^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\lambda\kappa} F_{\lambda\kappa} = u^\mu B^\nu_{(u)} - u^\nu B^\mu_{(u)}.$$

In the ideal MHD limit, the comoving and normal observer magnetic fields are related by $B^\mu_{(u)} = \frac{1}{W}(g^\mu{}_\nu + u^\mu u_\nu)B^\nu$, where $W \equiv -n_\mu u^\mu = \alpha u^t$ is the Lorentz factor.

With these definitions, the dual of the Faraday tensor is given by $*F^{\mu\nu} = \frac{1}{W}\left(u^\mu B^\nu - u^\nu B^\mu\right)$ and EM stress-energy tensor becomes

$T^{\mu\nu}_{\mathrm{EM}} = \left(u^\mu u^\nu + \frac{1}{2}g^{\mu\nu}\right)b^2 - b^\mu b^\nu$, where $b^\mu \equiv B^\mu_{(u)}$ and

$b^2 \equiv b^\mu b_\mu$.

Note that this definition depends on the units used, here we are using **Heaviside-Lorentz (HL) units**, which are rationalized (no appearance of explicit factors of 4π).

Assuming a **perfect fluid** (neglecting non-adiabatic effects such as explicit viscosity), the fluid stress-energy tensor is given by

$$T^{\mu\nu}_{\mathrm{matter}} = \rho h u^\mu u^\nu + P g^{\mu\nu}$$

And the **total stress-energy tensor** is

$$T^{\mu\nu} = \rho\, h^* u^\mu u^\nu + P^* g^{\mu\nu} - b^\mu b^\nu$$

Where $h^* \equiv 1 + \epsilon + (P + b^2)/\rho$ is the magnetically modified specific enthalpy and $P^* \equiv P + b^2/2$ is the magnetically modified isotropic pressure.

In principle we could evolve the covariant 4D evolution equations (see for instance the **HARM** family of GRMHD codes [Gammie et al 2003, Noble et al 2009, 2012]) but we will instead bring the equations in a 3+1 form to be compatible with the dynamic spacetime evolution (see **Helvi's lecture**).

The next step is to derive the equations of **ideal GRMHD** in a **3+1 split**…

3+1 split of the GRMHD equations: the Valencia formulation
[Banyuls et al 1997, Anton et al 2006]

Starting with the continuity equation and using the following identity $\nabla_\nu V^\nu = \frac{1}{\sqrt{|g|}}\partial_\nu(\sqrt{|g|}V^\nu)$:

$$\nabla_\nu(\rho u^\nu) = \frac{1}{\sqrt{-g}}\partial_\nu(\sqrt{-g}\rho u^\nu) = \frac{1}{\sqrt{-g}}\left(\partial_t(\sqrt{-g}\rho u^t) + \partial_i(\sqrt{-g}\rho u^i)\right)$$

$$= \partial_t(\sqrt{\gamma}\rho W) + \partial_i\left(\alpha\sqrt{\gamma}\rho W\bar{v}^i\right) = 0,$$

where $\bar{v}^i \equiv v^i - \frac{\beta^i}{\alpha} = \frac{u^i}{W}$ is the so-called **advection velocity** and

$$v^i \equiv \frac{1}{W}\gamma^i{}_\nu u^\nu = \frac{u^i}{W} + \frac{\beta^i}{\alpha}$$

is the **Valencia three-velocity,** and and we have rewritten $\sqrt{-g} = \alpha\sqrt{\gamma}$, where $\sqrt{\gamma}$ is the determinant of the three-metric γ_{ij} .

Next, we perform a **spatial projection** of the equation of stress-energy conservation and use the following identity:

$$\nabla_\lambda T^\lambda_\mu = \frac{1}{\sqrt{|g|}}\partial_\lambda(\sqrt{|g|}\,T^\lambda_\mu) - T^\lambda_\sigma \Gamma^\sigma_{\lambda\mu}$$

to arrive at the **evolution equation for the conserved momenta** (see e.g. [Montero et al 2014]):

$$\begin{aligned}\gamma_{i\nu}\nabla_\mu T^{\mu\nu} &= g_{i\nu}\nabla_\mu T^{\mu\nu} = \nabla_\mu(g_{i\nu}T^{\mu\nu}) \\ &= \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}T^\mu_i) - T^\nu_\mu\Gamma^\mu_{i\nu} = \partial_t(\alpha\sqrt{\gamma}T^t_i) + \partial_j(\alpha\sqrt{\gamma}T^j_i) - \alpha\sqrt{\gamma}T^\nu_\mu\Gamma^\mu_{i\nu} \\ &= \partial_t(\alpha\sqrt{\gamma}T^t_i) + \partial_j(\alpha\sqrt{\gamma}T^j_i) - \alpha\sqrt{\gamma}T^{\mu\nu}(\partial_\mu g_{\nu i} + \Gamma^\sigma_{\mu\nu}g_{\sigma i}) = 0.\end{aligned}$$

To arrive at an evolution equation for the conserved energy that recovers the correct Newtonian limit and is numerically more accurate than evolving the total conserved energy density, we **project the conservation of stress-energy along n_ν** and **subtract the continuity equation**:

$$
\begin{aligned}
\nabla_\mu\left(-n_\nu T^{\nu\mu} - \rho u^\mu\right) + T^{\mu\nu}\nabla_\nu n_\mu &= \nabla_\mu\left(\alpha T^{0\mu} - \rho u^\mu\right) + T^{\mu\nu}\nabla_\nu n_\mu \\
&= \partial_t\left(\alpha\sqrt{\gamma}(\alpha T^{00} - \rho u^t)\right) \\
&+ \partial_i\left(\alpha\sqrt{\gamma}(\alpha T^{0i} - \rho u^i)\right) + \alpha\sqrt{\gamma}\, T^{\mu\nu}\nabla_\nu n_\mu = 0
\end{aligned}
$$

where the source term can be expanded as follows:

$$
\alpha\sqrt{\gamma}\, T^{\mu\nu}\nabla_\nu n_\mu = \alpha\sqrt{\gamma}(T^{00}(\beta^i\beta^j K_{ij} - \beta^i\partial_i\alpha) + T^{0i}(2\beta^j K_{ij} - \partial_i\alpha) + T^{ij}K_{ij})
$$

Contrary to the continuity equation, we see that there are **source terms** appearing in the **momentum and energy equations**, reflecting the fact that the **fluid can interchange energy and momentum with the spacetime**.

Finally, using the following identity for antisymmetric tensors: $\nabla_\nu A^{\mu\nu} = \frac{1}{\sqrt{|g|}}\partial_\nu\left(\sqrt{|g|}A^{\mu\nu}\right)$, we can rewrite Maxwell's equations as follows:

$$\partial_t(\sqrt{\gamma}B^\mu) = \partial_i\left(\frac{\alpha\sqrt{\gamma}}{W}(u^\mu B^i - u^i B^\mu)\right).$$

As B^μ is purely spatial $\left(n_\mu B^\mu = 0 \rightarrow B^0 = 0\right)$ the temporal component of the Maxwell equation results in the solenoidal constraint for the magnetic field:

$$\partial_i\left(\sqrt{\gamma}B^i\right) = 0,$$

while the spatial components result in the induction equation:

$$\partial_t(\sqrt{\gamma}B^i) = \partial_j\left(\sqrt{\gamma}(\bar{v}^i B^j - \bar{v}^j B^i)\right).$$

Next, we will define our **conserved variables** and bring the system in a **first-order, flux-conservative** form…

The evolution equations as a first-order, flux-conservative hyperbolic system

[Banyuls et al 1997, Anton et al 2006]

Defining the following **conserved** variables:

$$D \equiv \sqrt{\gamma}\rho W, \qquad S_i \equiv \sqrt{\gamma}\left(\rho h^* W^2 v_j - \alpha b^0 b_j\right),$$

$$\tau \equiv \sqrt{\gamma}\left(\rho h^* W^2 - P^* - (\alpha b^0)^2\right) - D, \qquad \mathscr{B}^i \equiv \sqrt{\gamma} B^i,$$

where

$$b^0 = \frac{W B^i v_i}{\alpha},$$

$$b^i = \frac{B^i}{W} + W(B^j v_j)\bar{v}^i,$$

$$b^2 = \frac{B^i B_i}{W^2} + (B^i v_i)^2$$

we can write the 3+1 GRMHD evolution equations as a first order hyperbolic system that is suited for numerical integration.

The first-order, flux-conservative hyperbolic system for GRMHD

[Banyuls et al 1997, Anton et al 2006]

$$\partial_t D + \partial_i(\alpha D\, \bar{v}^i) = 0,$$

$$\partial_t S_j + \partial_i \left(\alpha (S_j \bar{v}^i + \sqrt{\gamma}\, P^* \delta^i_j - b_j \mathscr{B}^i / W) \right) = \alpha \sqrt{\gamma} T^{\mu\nu} (\partial_\mu g_{\nu i} + \Gamma^\sigma_{\mu\nu}\, g_{\sigma i}),$$

$$\partial_t \tau + \partial_i \left(\alpha (\tau \bar{v}^i + \sqrt{\gamma}\, P^* v^i - \alpha b^0 \mathscr{B}^i / W) \right) = \alpha \sqrt{\gamma} (T^{00} (\beta^i \beta^j K_{ij} - \beta^i \partial_i \alpha)$$
$$+ T^{0i} (2\beta^j K_{ij} - \partial_i \alpha) + T^{ij} K_{ij}),$$

$$\partial_t (\mathscr{B}^i) - \partial_j \left(\bar{v}^i \mathscr{B}^j - \bar{v}^j \mathscr{B}^i \right) = 0$$

Which we combine in the following compact notation as

$$\partial_t \mathbf{U}_A + \partial_i \mathbf{F}^i_A = \mathbf{S}_A,$$

where the index A indicates the field in the system of equations.

In systems of nonlinear hyperbolic PDEs such as the 3+1 GRMHD evolution system, **smooth initial data can develop discontinuities in the variables in finite time**. The reason the the original 4D conservation laws were rewritten as a first-order, flux-conservative system evolution system is that, in such a form, a numerical scheme that converges **guarantees the correct Rankine-Hugoniot conditions across discontinuities, which is called the shock-capturing property**. This property is at the heart of **high-resolution shock-capturing (HRSC)** methods that guarantee that the physics of the flow will be correctly modeled by the numerical scheme in the presence of discontinuities in the fluid variables.

$$\partial_t D + \partial_i(\alpha D\, \bar{v}^i) = 0$$

Approximating the divergence directly using finite differences or spectral methods will break down once discontinuities and shocks develop in the solution (in a single neutron star, the stellar surface presents such a problematic region already in the initial data).

Instead, we will integrate the GRMHD equations in a **finite volume formulation**….

We begin by integrating the system of equations over each cell:

$$\int_{\Delta V} \left(\partial_t \mathbf{U}_A\right)\ d^3x + \int_{\Delta V} \left(\partial_i \mathbf{F}^i_A\right)\ d^3x = \int_{\Delta V} \mathbf{S}_A\ d^3x$$

and use the divergence theorem to turn the volume integral of the flux divergence into a surface integral:

$$\partial_t \langle \mathbf{U}_A \rangle + \frac{1}{\Delta V} \oint_{\Delta A} \left(\mathbf{F}^i_A n_i\right)\ dA = \langle \mathbf{S}_A \rangle,$$

where $\langle T \rangle = \dfrac{1}{\Delta V} \displaystyle\int_{\Delta V} T\ d^3x$ and n_i is the surface normal to the cell face.

So far, the integration is exact, and we haven't made any approximations yet.

In this form, we see that (in the absence of source terms) the rate of change of the cell average of $\mathbf{U}_A$ is given by the total fluxes $\mathbf{F}^i_A$ though the cell faces bounding it.

Next, we will need to:

- Choose how we do volume and surface integrals.
- **Obtain the fluxes $\mathbf{F}^i_A$** at the cell faces which is at the heart of **HRSC schemes**.

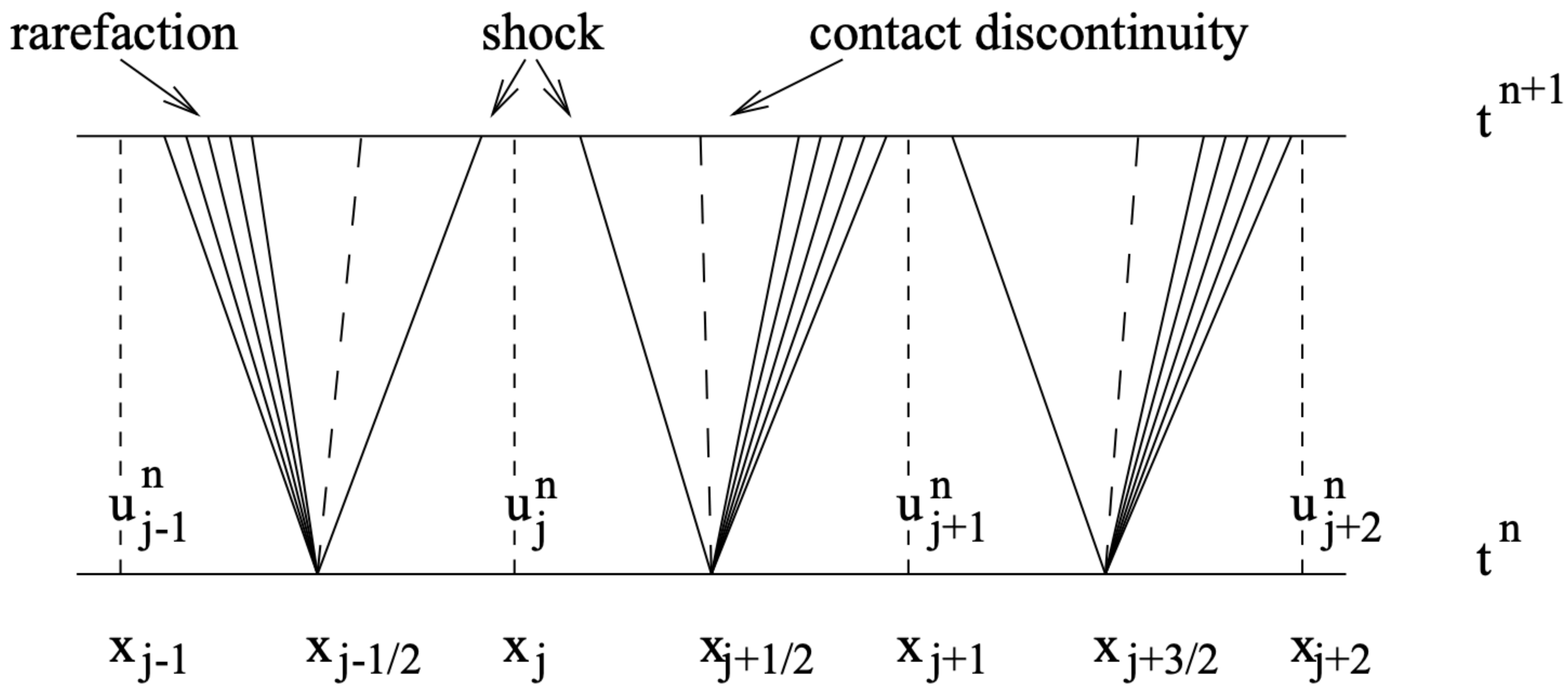
Using the **midpoint rule to perform volume and surface integrals**, which is a **second order accurate** approximation, the cell average $\langle \mathbf{U}_A \rangle$ is given by the value at the centre of the cell:

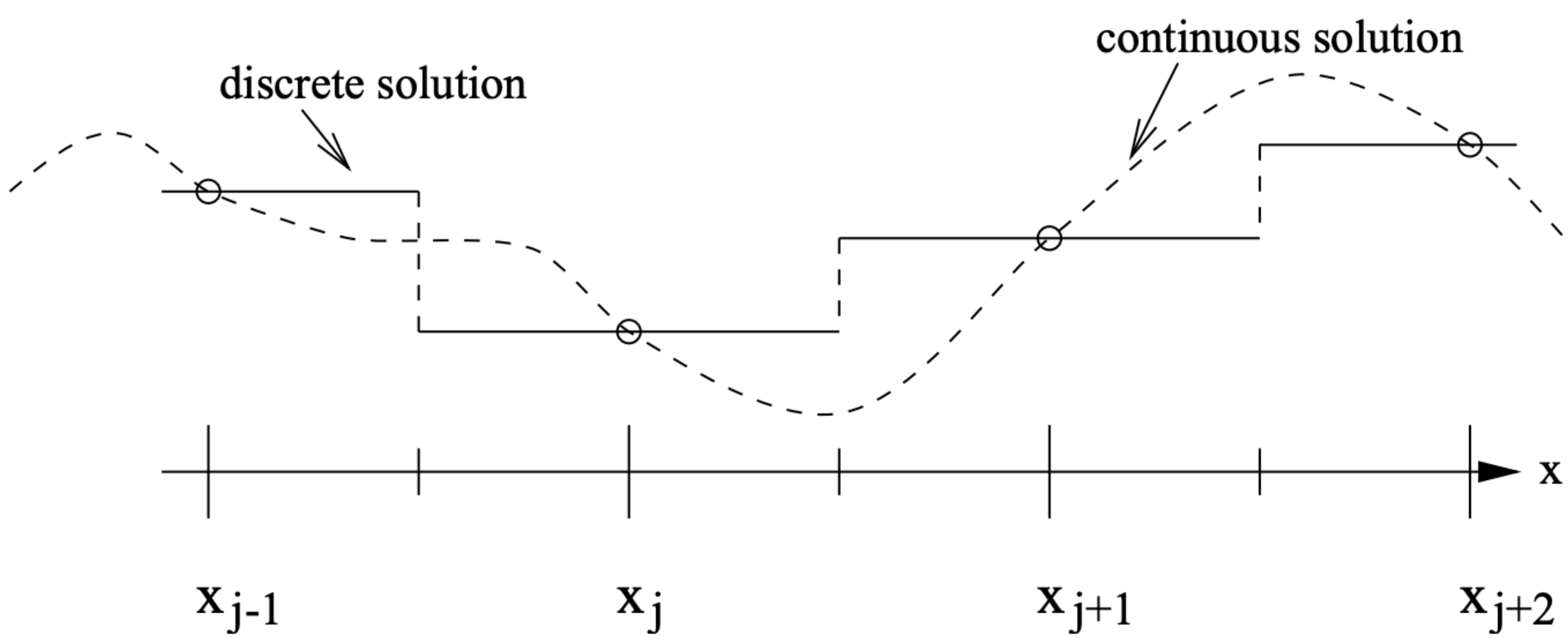
$$\langle \mathbf{U}_A \rangle = \mathbf{U}_{A(i,j,k)}$$

And similarly, the surface integrals of the fluxes $\mathbf{F}^i_A$ will be approximated by their value at the centre of the cell faces.

In order to estimate the face-centered fluxes $\mathbf{F}^i_A$, we need to:

- **Recover** the primitive variables $(\rho, \epsilon, v^i, B^i)$ from the conserved evolved variables $(D, S_i, \tau, \mathscr{B}^i)$, which in GRMHD involves root-finding [see e.g. Noble et al 2006], see **Wolfgang's talk**.
- **Reconstruct** the primitive variables at the face centres from both sides in a non-oscillatory manner.
- Solve a **Riemann problem** using the left and right state of the reconstructed primitives to get $\mathbf{F}^i_A$.





[From: J.A. Font, Living Reviews in Relativity 11 (1), 1-13, https://creativecommons.org/licenses/by/4.0]

Reconstruction: example using **TVD** (total variation diminishing) reconstruction using a **minmod limiter**:

For the cell face at $(i+\frac{1}{2}, j, k)$ we reconstruct the left and right states as

$$u^L_{i+\frac{1}{2},j,k} = u_{i,j,k} + \frac{1}{4}\left(\text{sign}(u_{i,j,k} - u_{i-1,j,k}) + \text{sign}(u_{i+1,j,k} - u_{i,j,k})\right) \min\left(\,|u_{i,j,k} - u_{i-1,j,k}|, |u_{i+1,j,k} - u_{i,j,k}|\,\right)$$
$$u^R_{i+\frac{1}{2},j,k} = u_{i+1,j,k} - \frac{1}{4}\left(\text{sign}(u_{i+1,j,k} - u_{i,j,k}) + \text{sign}(u_{i+2,j,k} - u_{i+1,j,k})\right) \min\left(\,|u_{i+1,j,k} - u_{i,j,k}|, |u_{i+2,j,k} - u_{i+1,j,k}|\,\right)$$

Other, more sophisticated reconstruction methods of higher order are:

- **PPM** (piecewise-parabolic method) [Colella and Woodward 1984]
- **MP5** (5th order monotonicity-preserving reconstruction) [Suresh and Huynh 1997]
- **ENO** (essentially non-oscillatory) reconstruction [Harten, Engquist, Osher, and Chakravarthy 1997]
- **WENO** (weighted essentially non-oscillatory) reconstruction [Shu 1998]
- **WENO-Z** (weighted essentially non-oscillatory)-Z reconstruction [Castro, Costa, and Don 2011]

Hyperbolic PDEs have a **finite propagation speed**, information can only travel at most at the **fastest characteristic speed of the system**.

The characteristic structure is given by the **eigenvalues of the Jacobian** matrices of the system:

$$\mathbf{J}^i = \frac{\partial \mathbf{F}^i_A}{\partial \mathbf{U}_A}$$

The characteristic structure of the Valencia formulation of GRMHD has been presented in [Anton et al 2006]. In classical MHD, there are **7 physical waves** [Brio and Wu 1988] and corresponding **characteristic speeds**:

- 2 Alfven waves with eigenvalues $\lambda_{a\pm} = v_x \pm v_a$
- 2 fast magnetosonic waves with eigenvalues $\lambda_{f\pm} = v_x \pm v_f$
- 2 slow magnetosonic waves with eigenvalues $\lambda_{s\pm} = v_x \pm v_s$
- 1 entropy wave with eigenvalue $\lambda_e = v_x$

The eigenvalues are ordered as follows:

$$\lambda_{f-} \leq \lambda_{a-} \leq \lambda_{s-} \leq \lambda_e \leq \lambda_{s+} \leq \lambda_{a+} \leq \lambda_{f+}$$

In GRMHD, obtaining the magnetosonic waves require the solution of a quartic equation [Anile 1990], while the entropy and Alfven waves are explicit.

Once we have reconstructed the primitives $(\rho, \epsilon, v^i, B^i)$ to their left and right states and have gotten spacetime variables $(\alpha, \beta^i, \gamma_{ij})$ on the cell faces, we can proceed with the final step:

Solving Riemann problems between the left and right states to approximate the fluxes $\mathbf{F}^i_A$.

The **HLLE** (Harten-Lax-van Leer- Einfeldt) approximate Riemann solver [Einfeldt 1988, Harten 1983] uses a **two-wave approximation** to obtain the solution to the Riemann problem at the cell faces, resulting in the following numerical fluxes (see e.g. the GRHydro paper [Moesta et al 2014]):

$$\tilde{\mathbf{F}}^i_A = \frac{\tilde{\lambda}^i_+ \mathbf{U}^L_A - \tilde{\lambda}^i_- \mathbf{U}^R_A + \tilde{\lambda}^i_+ \tilde{\lambda}^i_- \left(\mathbf{U}^R_A - \mathbf{U}^L_A\right)}{\tilde{\lambda}^i_+ - \tilde{\lambda}^i_-},$$

where $\tilde{\lambda}^i_+ = \max\left(0, \lambda^i_\pm\right)$ and $\tilde{\lambda}^i_- = \min\left(0, \lambda^i_\pm\right)$ $\qquad \lambda^i_\pm = (\lambda'^i_-, v^i - \dfrac{\beta^i}{\alpha}, \lambda'^i_+)_-, (\lambda'^i_-, v^i - \dfrac{\beta^i}{\alpha}, \lambda'^i_+)_+)$

$\lambda'^i_\pm$ can be estimated as (see, e.g. [Anton et al 2006, Porth et al 2017]):

$$\lambda'^i_\pm = \left((1 - a^2) v^i \pm \sqrt{a^2 (1 - v^2)[(1 - v^2 a^2)\gamma^{ii} - (1 - a^2)(v^i)^2]} \right) / (1 - v^2 a^2)$$

$$a^2 = c_s^2 + v_a^2 - c_s^2 v_a^2, \quad v_a^2 = \frac{b^2}{\rho h + b^2}$$

Updating the evolution in time using the Method of lines

After all these steps, having obtained an approximation for the intercell fluxes $\tilde{F}^i_A$, we are now able to integrate the evolution equations forward in time using the **Method of Lines**: having discretized our first-order hyperbolic system with the finite volume method in the spatial dimension, we integrate the equations in time using integrators for ODEs such as **Runge-Kutta methods** using the thorn **MoL**.

$$\partial_t \mathbf{U}_A = \mathrm{RHS}_A(\mathbf{U}_A, \mathbf{F}_A, \mathbf{S}_A)$$

$$\mathbf{U}_A^{n+1} = \mathbf{U}_A^n + \Delta t\, \mathrm{RHS}_A(\mathbf{U}_A, \mathbf{F}_A, \mathbf{S}_A) \quad \text{Euler step}$$

For the time integration, it is useful to chose **strong stability-preserving Runge-Kutta methods** for their TVD property [Shu and Osher 1988].

As an example, in Cartesian coordinates and our second order finite volume scheme, the RHS of the continuity equation is given by:

$$\partial_t D = \frac{\tilde{F}^x_{D,i-\frac{1}{2},j,k} - \tilde{F}^x_{D,i+\frac{1}{2},j,k}}{dx} + \frac{\tilde{F}^y_{D,i,j-\frac{1}{2},k} - \tilde{F}^y_{D,i,j+\frac{1}{2},k}}{dy} + \frac{\tilde{F}^z_{D,i,j,k-\frac{1}{2}} - \tilde{F}^z_{D,i,j,k+\frac{1}{2}}}{dz}$$

Recap

The Valencia formulation brings the evolution equations of GRMHD in a first-order, flux-conservative hyperbolic system.

Using a second-order accurate finite volume scheme where the cell-average of a field is equal to its value at the cell centre, the following steps are necessary to advance the evolution equations a timestep using the method of lines:

- **Recover** the primitive from the conserved variables at the cell centers.
- **Reconstruct** the primitives to the cell faces as two L/R biased states.
- Estimate the fastest and slowest **characteristic speeds**, and use them in an **approximate Riemann** solver to approximate the **numerical intercell fluxes**.
- Combine the surface integrals of the numerical fluxes and volume integrals of possible source terms together to form the **RHS**.
- Update the conserved variables in the cell centers forward in time using the **RHS and the Method of Lines**.

And we are done! Or are we…?

The induction equation and the solenoidal constraint

As indicated by its color, the induction equation, while being part of the evolution system, is much more delicate to handle correctly than all the other equations in the hyperbolic system:

$$\partial_t(\mathscr{B}^i) - \partial_j\left(\bar{v}^i\mathscr{B}^j - \bar{v}^j\mathscr{B}^i\right) = 0$$

If we were to follow the outlined steps, and integrate the induction equation just like all the others in a finite volume scheme using the methods of lines, the problem is that the solenoidal constraint $\partial_i\left(\sqrt{\gamma}B^i\right) = 0$, if it was fulfilled in the initial data, will obtain a truncation error level violation at every timestep, which eventually leads to unphysical forces and spoils the evolution [Brackbill and Barnes 1980].

In order to evolve the magnetic field in such a way that the solenoidal constraint is fulfilled, there are three different methods that are employed in GRMHD codes:

- Hyperbolic **divergence cleaning** using generalized Lagrange multipliers [Dedner et al 2002]: Evolve an auxiliary equation that transports violations to the solenoidal constraint out of the computational domain and damps them.
- **Constrained transport** [Evans and Hawley 1988, see also Toth 2000]: The magnetic field does not live at the cell-centers, but rather at the cell faces and is updated in such a way that its divergence remains constant in time. If one starts with initial data that fulfill the solenoidal constraint, it will be subsequently remain so.
- Evolution of the **vector potential** of the magnetic field (see, e.g. WhiskyMHD [Giacomazzo and Rezzolla 2007], IllinoisGRMHD [Etienne et al 2015], Spritz [Cipolletta et al 2019] codes): Instead of evolving the magnetic field directly, we can evolve its vector potential in a suitable gauge and obtain the magnetic field as the curl of the vector potential, which will automatically guarantee the solenoidal constraint.

As an example, evolving the vector potential in the generalized Lorenz gauge $\nabla_\mu A^\mu = -\zeta\Phi$ [Farris et al 2012] amounts to replacing the induction equation with the following evolution equations for the vector and EM scalar potential, as well as the update of the magnetic field via the curl of the vector potential:

$$\partial_t A_i = \alpha\,[ijk]\,\bar{v}^j\,\mathscr{B}^k - \partial_i\left(\alpha\Phi - \beta^j A_j\right)$$

$$\partial_t(\sqrt{\gamma}\,\Phi) + \partial_i\left(\alpha e^{6\phi}\sqrt{\gamma}\,A^i - \sqrt{\gamma}\beta^i\Phi\right) = -\zeta\alpha\sqrt{\gamma}\Phi$$

$$B^i = \epsilon^{ijk}\partial_j A_k$$

where $\epsilon^{ijk} \equiv [ijk]/\sqrt{\gamma}$.

Note that in the ideal GRMHD limit, $E_i = -\epsilon_{ijk}\bar{v}^j B^k$, where $\epsilon_{ijk} \equiv [ijk]\sqrt{\gamma}$,

So the that $\partial_t A_i = -\alpha E_i - \partial_i\left(\alpha\Phi - \beta^j A_j\right)$

In an un-staggered implementation, we can use higher order finite difference methods for the evolution equations of A_i, Φ and in the calculation of the curl of A_i (see **SphericalNR** talk on Friday).

One last stumbling block: the atmosphere

As a final subtlety in GRMHD, we note that the evolution equations are not apt to evolve true vacuum, as they break down there.

In absence of complicated numerical methods such as using moving, time-dependent fluid boundaries that coincide with the surface of a star, say, it is customary to use a tenuous atmosphere anywhere in the grid that is not vacuum and flooring the density, internal energy and pressure whenever they fall below their floor values.

The evolution of the magnetic field in the atmosphere is particularly difficult, as even weak magnetic fields may have a magnetic pressure that is orders of magnitude stronger than the fluid pressure in the atmosphere, which is one of the situations when the primitive recovery is particularly prone to fail.

GRHydro

In the final part, we will go through the relevant building blocks of a finite volume GRMHD code in the Valencia formulation covered in this lecture by taking a closer look at the **GRHydro thorn in the ETK**.