26 July 2021

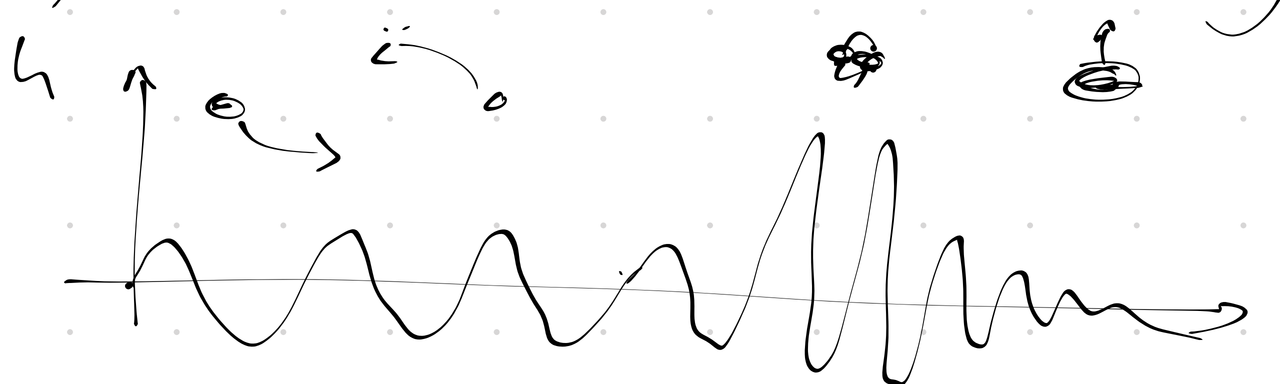
Einstein Toolkit Workshop & School

Introduction to Numerical Relativity

- Textbooks: · M. Alcubierre, "Introduction to 3+1 NR" (2008)
  - T. Baumgarte & S. Shapiro, "Numerical Relativity" (2010
    "Numerical Relativity: Starting from scratch" (2021)
  - M. Shibata, "Numerical Relativity" (2015)
- What is NR: – solve Einstein's Eqs in 3+1 dims as time evol. problem
  – typ. on HPC

- Applications:

i) Grav. wave source modelling



$v/c \ll 1$ PN   NR   perturbation

- groundbased detector: BH + BH
  NS + NS
  NS + BH
- spacebased detectors: – BH + BH
  – EMRI [self-force)
  – stochastic backgrounds

ii) BHs + light fields (e.g. dark matter)

iii) Strong field tests of gravity

iv) accretion disk, jet formation

v) cosmology → inflation
  → cosmic strings

vi) higher dimensions

vii) AdS / CFT

viii) BH stability

"Ingredients":

1) theoretical model:
today Einstein's Eqs in vac, in 4D asymptotically flat.

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0$$

2) Spacetime decomposition ←
3) 3+1 decomp for field eqs ←
(+ wellposedness)
4) Initial conditions ←
←
5) Gauge choices
6) Phys. observables, e.g. • Newman Penrose Ψ_4
→ strain
• apparent or isolated horizons.

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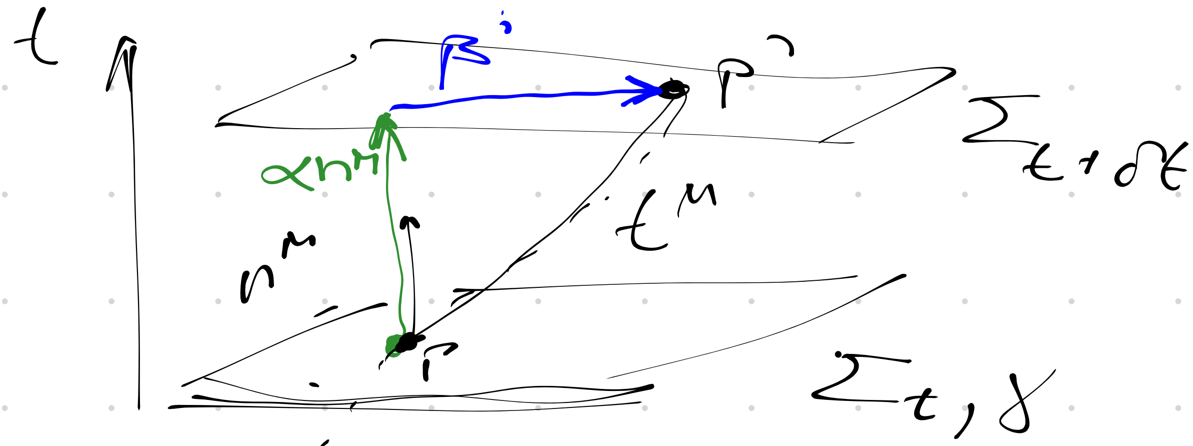
1) 3+1 decomposition of spacetime

- $\mu, \nu, \ldots = 0, \ldots, 3$
- $i, j, \ldots = 1, 2, 3$

↳ make time dependence explicit
↳ foliate 4D manifold $(\mathcal{M}, g)$ into spacelike hypersurfaces (Σ_t, γ) with level sets t



• on Σ_t: induced, spatial metric γ
measures proper distance in Σ: $dl^2 = \gamma_{ij} dx^i dx^j$

• timelike normal vector n^μ, s.t., $n^\mu n_\mu = -1$

• α: lapse, proper time between hypersurfaces for observer moving along n^μ

• β^i: shift vector: rel. velocity between normal obser and const spat. coords

• time vector $t^\mu = \alpha n^\mu + \beta^\mu$; $\beta^\mu n_\mu = 0$
(by constr.)

- relation $g_{\mu\nu} \leftrightarrow \gamma_{ij}, \alpha, \beta^i$:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$
$$= -(\alpha^2 - \gamma_{ij}\beta^i\beta^j)\, dt^2 + 2\gamma_{ij}\beta^i dt\, dx^j + \gamma_{ij} dx^i dx^j$$

Note: - $\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$

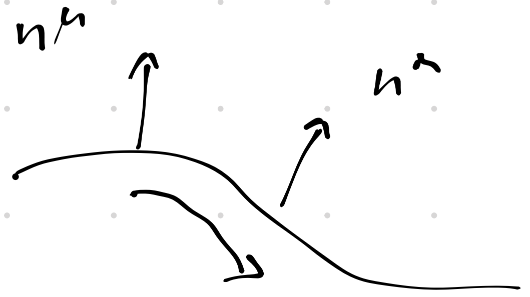
- γ defines a proj. op.: $\gamma^\mu{}_\nu = \delta^\mu{}_\nu + n^\mu n_\nu$
- any vector V^μ can be decomposed into a normal comp. $\mathcal{N} = -V^\mu n_\mu$ & spatial comp $\mathcal{V}^i = \gamma^i{}_\mu V^\mu$

$\rightarrow$ $V^\mu = \mathcal{N} n^\mu + \mathcal{V}^\mu$

- cov. deriv w.r.t γ_{ij}: $D_i = (\gamma \nabla)_i$
- Ricci (intrinsic) curvature:

$$(D_m D_n - D_n D_m) V^k = R^k{}_{lmn} V^l$$

- Extrinsic curvature K_{ij}:



- parallel transport n^μ along Σ_t
- extrinsic curvature: measure for the change of n^μ

$$\boxed{K_{\mu\nu} = -\gamma_\mu{}^\kappa \nabla_\kappa n_\nu}$$

show:
$$K_{\mu\nu} = -\frac{1}{2\alpha}\mathcal{L}_n \gamma_{\mu\nu} = -\frac{1}{2\alpha}(\partial_t - \mathcal{L}_\beta)\gamma_{\mu\nu}$$

$\rightarrow$ $K_{\mu\nu}$ - as "momentum" of metric

$\rightarrow$ $\boxed{\partial_t \gamma_{ij} = -2\alpha K_{ij} + \mathcal{L}_\beta \gamma_{ij}}$ kinematics

Summarize: describe ST in terms of: $(\gamma_{ij}, \alpha, \beta^i)$

- $K_{ij} \sim \partial_t \gamma_{ij}$

2) 3+1 decomposition of field equations

- dynamics
- need: (i) projections of Riemann tensor

$$ {}^{(4)}R_{\mu\nu\rho\sigma} \longleftrightarrow {}^{(3)}R_{\mu\nu\rho\sigma} $$

∟ Gauss - Codazzi relations

(ii) projections of energy-momentum tensor $T_{\mu\nu}$

∟ energy density $\rho = T_{\mu\nu} n^\mu n^\nu$

∟ energy-mom flux $j_i = -\gamma^\mu_i n^\nu T_{\mu\nu}$

∟ stress tensor $S_{ij} = \gamma_i^{\ \mu} \gamma_j^{\ \nu} T_{\mu\nu}$

(iii) projection of field equations
here $G_{\mu\nu} = 0$

$\rightarrow$

A) constraints

∟ $\mathcal{H} = 2 G_{\mu\nu} n^\mu n^\nu = 0 = {}^{(3)}R - K_{ij} K^{ij} + K^2$ Hamiltonian

∟ $\mathcal{M}_i = -\gamma^\mu_i n^\nu G_{\mu\nu} = D^j K_{ij} - D_i K = 0$ momentum constraint

$[K = \gamma^{ij} K_{ij}]$

∟ elliptic equations

∟ in free evol scheme: solve $\mathcal{H}, \mathcal{M}_i$ for initial data

- monitor $\mathcal{H}, \mathcal{M}_i$ during evolution
- Bianchi ids. imply that $\mathcal{H} = 0$, $\mathcal{M}_i = 0$ during evolution if satisfied @ $t=0$

B) Evolution equations

∟ project $G_{\mu\nu} \gamma^\mu_i \gamma^\nu_j = 0 \Rightarrow \mathcal{L}_n K_{ij}$

dynamics $\longrightarrow$
$$ (\partial_t - \mathcal{L}_\beta) K_{ij} = - D_i D_j \alpha + \alpha [R_{ij} + K K_{ij} - 2 K_i^{\ k} K_{kj}] $$

kinematic
$$ (\partial_t - \mathcal{L}_\beta) \gamma_{ij} = -2\alpha K_{ij} $$

ADM - York equations
(Arnowitt - Deser - Misner)

3) Wellposedness of evol. eqs.

Def.: A system of PDE

$$\begin{cases} \partial_t f = A^P \partial_p f + Bf \\ f(t=0) = h \end{cases}$$

- f - vector of variables
- ∂_p - spatial derivs
- A^P - principal symbol

is said to be well-posed if
- $\exists$ a unique solution that depends continuously on smooth data

In particular, a system is well-posed if $\exists\, k = \text{const}$, $a = \text{const}$

s.t. $\|f(t,.)\| \le k e^{at} \|f(t=0,.)\|$

↳ lay-person version: recover a set of wave eqs.
↳ heuristic version

4D) $G_{\mu\nu} = 0 = \underbrace{g^{\kappa\lambda}\partial_\kappa\partial_\lambda g_{\mu\nu}}_{\Box g_{\mu\nu}} + \text{"}g^{\kappa\lambda}\partial_\mu\partial_\kappa g_{\lambda\nu}\text{"}$

$+ (\partial g)^2 \ldots$

can spoil hyperbolicity

introduce harmonic gauge (Choquet-Bruhat)

$\longrightarrow \Box g_{\mu\nu} + \text{l.o.t.} \ldots$

Scalar wave eq.

"$\Box\phi = 0$"

3+1) leading order

$\partial_t \gamma_{ij} \simeq K_{ij}$

$\gamma_{ij} \sim \phi$
$K_{ij} \sim \Pi$

$\partial_t K_{ij} \simeq -D_i D_j \alpha + \alpha R_{ij} + \ldots$

$\simeq -D_i D_j \alpha$

$+ \alpha \Big[\underbrace{\gamma^{kl}\partial_k\partial_l \gamma_{ij}}_{\text{"}\Delta\gamma\text{"}} + \underbrace{\gamma^{kl}\partial_i\partial_k \gamma_{jm}}_{\text{cause for illposedness}} + \ldots$

introducing $\Pi = -\partial_t \phi$

$\partial_t \phi = -\Pi$

$\partial_t \Pi = -\underbrace{D^i D_i \phi}$

cure?
- introduce new variables
- add constraints

$\longrightarrow$ Baumgarte-Shapiro-Shibata-Nakamura (BSSN)
1999 / 1995

4) Gauge choices

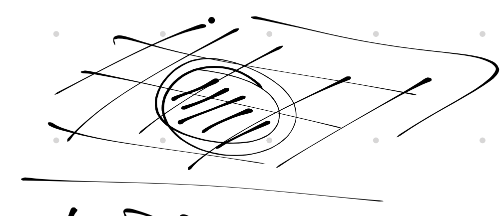
(ie, choice of α, β^i)

Note: simple is not always best!

e.g. $\alpha = 1, \beta^i = 0$ → reach singularity in finite time

Wishlist: – avoid reaching singularity

[– excision of inner region



[– clever choice of coords

– evol eqs + gauge form well-posed PDE

– easy to implement, efficient to evolve (avoid elliptic choices)

common: time evol for α, β^i

Here: punctive gauge

└ 1+log slicing for lapse α

Γ-driver eqn. for shift β^i

a) lapse: choice 1) maximal slicing: $K = 0$ (elliptic eqs)

insert in $\partial_t K_{ij}$, ⇒ $-\Delta\alpha + \alpha R = 0$

→ " $\alpha \sim e^{-R_0}$ "

→ α collapse to zero near singularity

spat. hypersurfaces cannot be arbitrarily close to singularity

→ "singularity avoidance"

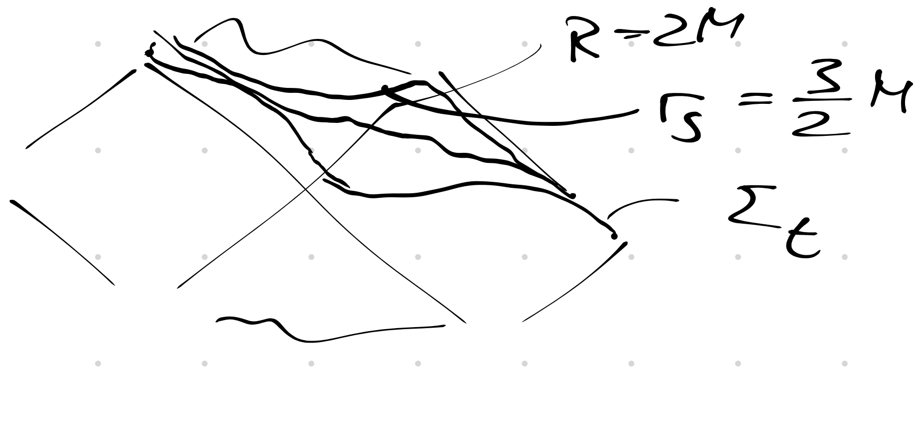
b) hyperbolic slicing conditions

general form (Bona – Masso)

$$\partial_t \alpha \simeq -\alpha^2 f(\alpha) (K - K_0)$$

for $f(\alpha) = \frac{2}{\alpha}$ ⇒ 1+log slicing

$$\Rightarrow \partial_t \alpha = -2\alpha (K - K_0)$$



c) shift: Γ-driver

$$\partial_t \beta^i = \beta_\Gamma \tilde{\Gamma}^i - \eta_\beta \beta^i$$

$\tilde{\Gamma}^i \simeq \partial_j \tilde{\gamma}^{ij}$ conformal conn. function

5) Initial data for black holes

∟ goal: $(\gamma_{ij}, K_{ij})\big|_{t=0}$

∟ Solve the Hamiltonian and momentum constraint (fixes 4 dofs)

+ physical / technical assumptions

a) Single, non-rotating, non-boosted BH (in 4D asympt. flat)

∟ time symmetric $K_{ij} = 0$

$\mathcal{M}_i = D^j K_{ij} - D_i K = 0$ ✓ trivially satisfied

∟ need to solve: Hamiltonian $\mathcal{H} = R - \underbrace{K_{ij}K^{ij} + K^2}_{=0}$
$= R = 0$

∟ metric ansatz

$$\rightarrow \gamma_{ij} = \psi^4 \tilde{\gamma}_{ij} = \psi^4 \eta_{ij}$$

conformal factor; conformal metric; conformally flat metric ansatz

∟ asymptotically flat ST: $\lim_{r \to \infty} \psi = 1$

insert in $\mathcal{H} = 0$: $\Delta_{flat} \psi = 0$

$\rightarrow$ solution: $\psi = 1 + \dfrac{k}{R}$

identify $k = M/2$

$$\Rightarrow \psi = 1 + \frac{M}{2R}$$

in metric: $ds^2 = -\alpha^2 dt^2 + \psi^4 \eta_{ij}\, dx^i dx^j$
Schwarzschild in isotropic coord.

b) for N-BHs w/o momenta
Laplacian is linear $\Rightarrow \psi = 1 + \sum_{A=1}^{N} \dfrac{m_{(A)}}{2|R - R_{(A)}|}$

bare mass param.; location

$\rightarrow$ Brill-Lindquist data

c) Bowen-York data [Brandt & Brügmann for N-BHs]

→ initial data for BHs with linear and angular momenta

- in Einstein Toolkit: implemented TwoPunctures thorn.